Mathematics Module 1 (Calculus and Statistics)

Solution	Marks	Remarks
1. (a) $k^2+0.16+0.18+0.3+k+0.12=1$ $k=0.2$ or $k=-1.2$ (rejected) $\therefore k=0.2$		
(b) $\mathrm{E}(X)$ $=(0)0.2^2+(2)0.16+(4)0.18+(5)0.3+(8)0.2+(9)0.12$ $=5.22$		
(c) $\mathrm{E}(X^2)$ $=(0)^2 0.2^2+(2)^2 0.16+(4)^2 0.18+(5)^2 0.3+(8)^2 0.2+(9)^2 0.12$ $=33.54$ $\mathrm{Var}(X)$ $=\mathrm{E}(X^2)-(\mathrm{E}(X))^2$ $=33.54-(5.22)^2$ $=6.2916$ $\mathrm{Var}(2-3X)$ $=9\mathrm{Var}(X)$ $=56.6244$		

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2. (a) $P(B\mid A)$ $=\dfrac{P(A\cap B)}{P(A)}$ $=\dfrac{P(A\mid B)P(B)}{P(A)}$ $=\dfrac{P(A\mid B)P(B')}{P(A)}$ $=\dfrac{P(A\mid B)\left(1-P(B)\right)}{P(A)}$ $=\dfrac{0.6\left(1-0.3\right)}{0.2}$ $=0.9$		
(b) Since $P(B\mid A)\neq 0$, A and B are NOT mutually inclusive.		
(c) $P(A)P(B)$ $=0.2(1-0.7)$ $=0.06$ $\neq P(A\cap B)$ So, A and B are NOT independent.		

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3. (a) Let X kg be the weight of chickens $X \sim N(\mu, \sigma^2)$ $\begin{cases} P(X < 1.83) = P(X > 3.43) \\ P(1.83 \le X \le 3.43) = 0.8904 \end{cases}$ $P(X < 1.83) + P(X > 3.43) = 1 - 0.8904$ $2P(X < 1.83) = 0.1096$ $P(X < 1.83) = 0.0548$ $P(Z < \frac{1.83 - \mu}{\sigma}) = 0.0548$ $P(0 \le Z < -\frac{1.83 - \mu}{\sigma}) = 0.4452$ $-\frac{1.83 - \mu}{\sigma} = 1.6$ $P(X > 3.43) = 0.0548$ $P(0 \le Z < \frac{3.43 - \mu}{\sigma}) = 0.4452$ $\frac{3.43 - \mu}{\sigma} = 1.6$ $\mu + 1.6\sigma = 3.43$ Solving, $\mu = 2.63$ and $\sigma = 0.5$		
(b) Let $\overline{X_9}$ kg be the mean weight of 9 chickens $\overline{X_9} \sim N(2.63, \frac{0.5^2}{9})$ Requited Probability $= P(2.5 \le \overline{X_9} \le 3.1)$ $= P(\frac{2.5 - 2.63}{\sqrt{\frac{0.5^2}{9}}} \le \overline{X_9} \le \frac{3.1 - 2.63}{\sqrt{\frac{0.5^2}{9}}})$ $= P(-0.78 \le \overline{X_9} \le 2.82)$ $= 0.2823 + 0.4976$ $= 0.7799$		

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4. (a) Required Probability $= 0.4^3 \times 0.6$ $= 0.0384$		
(b) $0.6 + 0.4 \times 0.6 + ... + 0.4^{9-k} \times 0.6 > 0.95$ $\sum_{x=0}^{9-k} 0.4^x \times 0.6 > 0.95$ $\frac{1-0.4^{10-k}}{1-0.4} \times 0.6 > 0.95$ $0.4^{10-k} < 0.05$ $10 - k > \frac{\ln 0.05}{\ln 0.4}$ $0 \le k < 6.730587608$ The greatest value of k is 6		
(c) The expected amount of money $= 15 \times \frac{1}{0.6}$ $= \$25$		
5. (a) $e^{3x} = 1 + 3x + \frac{(3x)^2}{2} + ...$ $e^{3x} = 1 + 3x + \frac{9x^2}{2} + ...$ $1 + e^{3x} = 2 + 3x + \frac{9x^2}{2} + ...$ $(1+e^{3x})^2$ $= \left(2 + 3x + \frac{9x^2}{2} + ...\right)^2$ $= 4 + 12x + 27x^2 + ...$		
(b) $(5-x)^4 = \sum_{k=0}^{4} C_k^4 5^{4-k} (-1)^k x^k$ The coefficient of x^2 $= 4 \times C_2^4 5^{4-2} (-1)^2 + 12 \times C_1^4 5^{4-1} (-1)^1 + 27 \times C_0^4 5^{4-0} (-1)^0$ $= 11475$		

Solution	Marks	Remarks
6. (a) $f(6)=-33$		
$4(6)^3+m(6)^2+n(6)+615=-33$		
$6m+n=-252$		
$f'(x)=12x^2+2mx+n$		
$f'(6)=0$		
$12(6)^2+2m(6)+n=0$		
$12m+n=-432$		
Solving, $m=-30$ and $n=-72$		
(b) $f'(x)=12x^2-60x-72$		
$f'(x)=0 \Leftrightarrow x=6$ or $x=-1$		

x	$(-\infty, -1)$	-1	$(-1, 6)$	6	$(6, +\infty)$
$f'(x)$	+	0	-	0	+

The minimum value of $f(x)$
$= f(6)$
$= -33$

The maximum value of $f(x)$
$= f(-1)$
$= 653$

Solution	Marks	Remarks
7. (a) $\dfrac{dy}{dx}$		
$=\dfrac{\sqrt{x-2}-x\frac{1}{2}(x-2)^{-\frac{1}{2}}}{x-2}$		
$=\dfrac{x-2-x\frac{1}{2}}{(x-2)^{\frac{3}{2}}}$		
$=\dfrac{x-4}{2(x-2)^{\frac{3}{2}}}$		
(b) Let (a, b) be the point of contact $(a > 2)$		
$\begin{cases} b=\dfrac{a}{\sqrt{a-2}} \\ \dfrac{b-0}{a-9}=\left.\dfrac{dy}{dx}\right\rvert_{(a,b)} \end{cases}$		
$\begin{cases} b=\dfrac{a}{\sqrt{a-2}} \\ \dfrac{b-0}{a-9}=\dfrac{a-4}{2(a-2)^{\frac{3}{2}}} \end{cases}$		
$\dfrac{a}{\sqrt{a-2}}=\dfrac{(a-4)(a-9)}{2(a-2)^{\frac{3}{2}}}$		
$2a(a-2)=(a-4)(a-9)$		
$a^2+9a-36=0$		
$a=3$ or $a=-12$ (rejected)		
The slope of tangent		
$=\left.\dfrac{dy}{dx}\right\rvert_{x=3}$		
$=-\dfrac{1}{2}$		

Solution	Marks	Remarks
8. (a) Let $y=\ln\left(\frac{e}{x}\right)$ $\frac{dy}{dx}=\frac{-\frac{e}{x^2}}{\frac{e}{x}}$ $dy=-\frac{1}{x}dx$ $\int g(x)dx$ $=\int\frac{1}{x}\ln\left(\frac{e}{x}\right)dx$ $=-\int y\,dy$ $=-\frac{y^2}{2}$ + constant $=-\frac{1}{2}\left(\ln\left(\frac{e}{x}\right)\right)^2$ + constant		
(b) (i) x-intercept of $\Gamma = 0$		
(ii) The required area $=\int_1^e g(x)dx-\int_e^{e^2} g(x)dx$ $=\left[-\frac{1}{2}\left(\ln\left(\frac{e}{x}\right)\right)^2\right]_1^e-\left[-\frac{1}{2}\left(\ln\left(\frac{e}{x}\right)\right)^2\right]_e^{e^2}$ $=1$		

Solution	Marks	Remarks
9. (a) (i) Estimate value of μ $=\frac{1}{40}\left(\frac{0.5+1.0}{2}(11)+\frac{1.0+1.5}{2}(13)+\frac{1.5+2.0}{2}(8)+\frac{2.0+2.5}{2}(5)+\frac{2.5+3.0}{2}(3)\right)$ $=1.45$ 90% confidence interval for μ $=\left(1.45-1.645\frac{0.4}{\sqrt{40}},1.45+1.645\frac{0.4}{\sqrt{40}}\right)$ $=(1.345961065, 1.554038935)$ $=(1.3460, 1.5540)$		
(ii) Suppose there are n samples $2\times 2.17\times\frac{0.4}{\sqrt{n}}\leq 0.3$ $n\geq 33.48551111$ The least sample size required is 34.		
(b) (i) Let T hours be the daily time spent T~N(1.48, 0.4^2) The required probability $=\mathrm{P}(T>2)$ $=\mathrm{P}(Z>\frac{2-1.48}{0.4})$ $=\mathrm{P}(Z>1.3)$ $=0.5-0.4032$ $=0.0968$		
(ii) The required probability $=\frac{C_1^9(1-0.0968)^8(0.0968)(0.0968)}{1-(1-0.0968)^{15}-C_1^{15}(1-0.0968)^{14}(0.0968)}$ $=0.086102962$ ≈ 0.0861 (corr to 4 d.p.)		

Solution	Marks	Remarks
10. (a) The required probability $=\sum_{k=0}^{4}\frac{e^{-2}2^k}{k!}$ $=0.947346982$ ≈ 0.9473 (corr to 4 d.p.)		
(b) $P(0\le x<500)$ $=1-0.45-0.2-0.1$ $=0.25$ The required probability $=\frac{e^{-2}2^3}{3!}\left(C_1^3(0.25)^2(0.1)+C_1^3(0.25)(0.2)^2+C_1^3(0.45)^2(0.2)\right)$ $=0.030721109$ ≈ 0.0307 (corr to 4 d.p.)		
(c) The required probability $=C_1^4(0.25)^3(0.1)+C_2^4(0.25)^2(0.2)^2+\frac{4!}{1!2!1!}(0.45)^2(0.25)(0.2)$ $+C_0^4(0.45)^4$ $=0.18375625$ ≈ 0.1838 (corr to 4 d.p.)		
(d) The required probability $=\dfrac{\frac{e^{-2}2^1}{1!}(0.1)+\frac{e^{-2}2^2}{2!}\left(C_1^2(0.25)(0.1)+0.2^2\right)+0.030721109+\frac{e^{-2}2^4}{4!}(0.18375625)}{\sum_{k=0}^{4}\frac{e^{-2}2^k}{k!}}$ $=0.104214881$ ≈ 0.1042 (corr to 4 d.p.)		

Solution	Marks	Remarks
11. (a) For Ada, I $=\int_{0.5}^{1} \mathrm{g}(x)\mathrm{d}x$ $\approx \frac{1}{2}\left(\frac{1-0.5}{5}\right)\left(\mathrm{f}(0.5)+2\mathrm{f}(0.6)+2\mathrm{f}(0.7)+2\mathrm{f}(0.8)+2\mathrm{f}(0.9)+\mathrm{f}(1)\right)$ $=0.747559671$ ≈ 0.7476 (corr to 4 d.p.) For Billy, I $=\int_{0.5}^{1} \mathrm{g}(x)\mathrm{d}x$ $\approx \int_{0.5}^{1} \frac{1+0.1x+0.005x^2}{x}\mathrm{d}x$ $=\int_{0.5}^{1}\left(\frac{1}{x}+0.1+0.005x\right)\mathrm{d}x$ $=\left[\ln\|x\|+0.1x+\frac{0.005}{2}x^2\right]_{0.5}^{1}$ $=0.74502218$ ≈ 0.7450 (corr to 4 d.p.)		

Solution	Marks	Remarks
(b) For Ada, $\mathrm{f}'(x)$ $=\dfrac{x\left(0.1e^{0.1x}\right)-e^{0.1x}}{x^2}$ $=e^{0.1x}\left(\dfrac{0.1}{x}-\dfrac{1}{x^2}\right)$ $\mathrm{f}''(x)$ $=e^{0.1x}\left(-\dfrac{0.1}{x^2}+\dfrac{2}{x^3}\right)+0.1e^{0.1x}\left(\dfrac{0.1}{x}-\dfrac{1}{x^2}\right)$ $=e^{0.1x}\left(\dfrac{0.01x^2-0.2x+2}{x^3}\right)$ $=e^{0.1x}\left(\dfrac{0.01(x-10)^2+1}{x^3}\right)$ For $0.5\le x\le 1$, $e^{0.1x}>0$, $x^3>0$ and $0.01(x-10)^2+1>0$ $\mathrm{f}''(x)>0$, for $0.5\le x\le 1$ $\mathrm{f}(x)$ is concave upward for $0.5\le x\le 1$ It is an over-estimate. For Billy, $e^{0.1x}=\displaystyle\sum_{k=0}^{\infty}\frac{(0.1x)^k}{k!}$ $e^{0.1x}-\left(1+0.1x+0.005x^2\right)=\displaystyle\sum_{k=3}^{\infty}\frac{0.1^k}{k!}x^k$ $\dfrac{e^{0.1x}}{x}-\dfrac{1+0.1x+0.005x^2}{x}=\displaystyle\sum_{k=2}^{\infty}\frac{0.1^k}{k!}x^k$ $\displaystyle\int_{0.5}^{1}\left(\frac{e^{0.1x}}{x}-\frac{1+0.1x+0.005x^2}{x}\right)\mathrm{d}x=\int_{0.5}^{1}\sum_{k=2}^{\infty}\frac{0.1^k}{k!}x^k\mathrm{d}x$ For $0.5\le x\le 1$ and k = 2, 3, …, $0.1^k>0$, $x^k>0$, $k!>0$ $\displaystyle\int_{0.5}^{1}\sum_{k=2}^{\infty}\frac{0.1^k}{k!}x^k\mathrm{d}x>0$ $\displaystyle\int_{0.5}^{1}\left(\frac{e^{0.1x}}{x}\right)\mathrm{d}x>\int_{0.5}^{1}\frac{1+0.1x+0.005x^2}{x}\mathrm{d}x$ It is an under-estimate. (c) $0.74502218\le I\le 0.747559671$ $-0.00097782\le I-0.746\le 0.001559672$ $0\le\lvert I-0.746\rvert\le 0.001559672<0.002$ Yes, I agree with the claim		

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12. (a) (i) $(x-4)(x-1)$ $=\left(4+\dfrac{3k}{2^{\lambda t}-k}-4\right)\left(4+\dfrac{3k}{2^{\lambda t}-k}-1\right)$ $=\left(\dfrac{3k}{2^{\lambda t}-k}\right)\left(\dfrac{3\times 2^{\lambda t}-3k+3k}{2^{\lambda t}-k}\right)$ $=\dfrac{9k2^{\lambda t}}{\left(2^{\lambda t}-k\right)^2}$		
(ii) For $t\ge 0$, $k>0$, $2^{\lambda t}>0$, $\left(2^{\lambda t}-k\right)^2>0$ $(x-4)(x-1)>0$, for $t\ge 0$ $x<1$ or $x>4$, for $t\ge 0$ Yes, I agree the claim		
(b) (i) $\dfrac{dx}{dt}$ $=-3k\dfrac{\lambda\ln 2}{\left(2^{\lambda t}-k\right)^2}$ $=-\dfrac{\lambda\ln 2}{3}\dfrac{9k}{\left(2^{\lambda t}-k\right)^2}$ $=-\dfrac{\lambda\ln 2}{3}(x-4)(x-1)$ So, we have $-\dfrac{\ln 2}{24}=-\dfrac{\lambda\ln 2}{3}$ $\lambda=\dfrac{1}{8}$		

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(ii) (1) When $t = 0$, $x = 0.8$ $0.8 = 4 + \frac{3k}{2^{\frac{1}{8}(0)} - k}$ $0.8(1-\mathrm{k}) = 4(1-\mathrm{k}) + 3k$ $k = 16$ Note that $\frac{\mathrm{d}x}{\mathrm{d}t} < 0$, x is a strict decreasing function of t. As $x > 0$ when $t = 0$, the crocodiles will be extinct only if $x = 0$ when $t > 0$ Put x = 0, we have $0 = 4 + \frac{48}{2^{\frac{1}{8}t} - 16}$ $2^{\frac{1}{8}t} = 4$ $t = 16$ So, the crocodiles will become extinct after 16 years elapsed since the start of the research.		
(ii) (2) When $t = 0$, $x = 7$ $7 = 4 + \frac{3k}{2^{\frac{1}{8}(0)} - k}$ $7(1-\mathrm{k}) = 4(1-\mathrm{k}) + 3k$ $k = \frac{1}{2}$ Put x = 0, we have $0 = 4 + \frac{1.5}{2^{\frac{1}{8}t} - 0.5}$ $2^{\frac{1}{8}t} = \frac{1}{8}$ $t = -24 < 0$ So, the crocodiles will **NOT** become extinct. The number of crocodiles after a long time $= \lim_{t \to \infty} \left(4 + \frac{1.5}{2^{\frac{1}{8}t} - 0.5} \right)$ $= \lim_{t \to \infty} \left(4 + \frac{3\left(2^{-\frac{1}{8}t}\right)}{2 - \left(2^{-\frac{1}{8}t}\right)} \right)$ $= 4 + \frac{3(0)}{2-(0)}$ $= 4$ thousands		