Mathematics Module 2 (Algebra and Calculus)

Solution	Marks	Remarks
1. $\frac{d}{d\theta}\sec 6\theta$ $=\lim_{h\to 0}\frac{\sec(6(\theta+h))-\sec 6\theta}{h}$ $=\lim_{h\to 0}\frac{\cos 6\theta-\cos(6(\theta+h))}{h\cos(6(\theta+h))\cos 6\theta}$ $=\lim_{h\to 0}\frac{-2\sin\frac{6\theta+6h+6\theta}{2}\sin\frac{6\theta-6h-6\theta}{2}}{h\cos(6(\theta+h))\cos 6\theta}$ $=\lim_{h\to 0}\frac{2\sin(6\theta+3h)\sin 3h}{h\cos(6(\theta+h))\cos 6\theta}$ $=\lim_{h\to 0}\frac{6\sin(6\theta+3h)}{\cos(6(\theta+h))\cos 6\theta}\frac{\sin 3h}{3h}$ $=\frac{6\sin 6\theta}{\cos 6\theta\cos 6\theta}(1)$ $=6\tan 6\theta\sec 6\theta$		

Solution	Marks	Remarks
2. $(1+ax)^8=\sum_{k=0}^{8}C_k^8a^kx^k$		
$(b+x)^9=\sum_{k=0}^{9}C_k^9b^{9-k}x^k$		
$\lambda_k=C_k^8a^k$ for $k=0, 1, 2, 3, \dots, 8$		
$\mu_k=C_k^9b^{9-k}$ for $k=0, 1, 2, 3, \dots, 9$		
$\frac{\lambda_2}{\mu_7}=\frac{7}{4}$		
$\frac{C_2^8a^2}{C_7^9b^{9-7}}=\frac{7}{4}$		
$4a^2=9b^2$		
$\lambda_1+\mu_8+6=0$		
$C_1^8a^1+C_8^9b^{9-8}+6=0$		
$8a+9b=-6$		
$b=\frac{-6-8a}{9}$		
$4a^2=9\left(\frac{-6-8a}{9}\right)^2$		
$-28a^2-96a-36=0$		
$a=-3$ or $a=-\frac{3}{7}$		

Solution	Marks	Remarks
3. (a) $\overrightarrow{OP}$ $=\dfrac{3\overrightarrow{OB}+2\overrightarrow{OA}}{3+2}$ $=\dfrac{2}{5}\mathbf{a}+\dfrac{3}{5}\mathbf{b}$		
(b) (i) $\mathbf{a}\cdot\mathbf{b}$ $=\mathbf{a}\cdot\mathbf{b}$ $=\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\angle AOB$ $=(45)(20)\left(\dfrac{1}{4}\right)$ $=225$		
(ii) $\left\lvert\overrightarrow{OP}\right\rvert^2$ $=\overrightarrow{OP}\cdot\overrightarrow{OP}$ $=\left(\dfrac{2}{5}\mathbf{a}+\dfrac{3}{5}\mathbf{b}\right)\cdot\left(\dfrac{2}{5}\mathbf{a}+\dfrac{3}{5}\mathbf{b}\right)$ $=\left(\dfrac{2}{5}\right)^2\mathbf{a}\cdot\mathbf{a}+\left(\dfrac{3}{5}\right)^2\mathbf{b}\cdot\mathbf{b}+\left(\dfrac{2}{5}\right)\left(\dfrac{3}{5}\right)\mathbf{a}\cdot\mathbf{b}+\left(\dfrac{2}{5}\right)\left(\dfrac{3}{5}\right)\mathbf{b}\cdot\mathbf{a}$ $=\left(\dfrac{2}{5}\right)^2\lvert\mathbf{a}\rvert^2+\left(\dfrac{3}{5}\right)^2\lvert\mathbf{b}\rvert^2+(2)\left(\dfrac{2}{5}\right)\left(\dfrac{3}{5}\right)\mathbf{a}\cdot\mathbf{b}$ $=\left(\dfrac{2}{5}\right)^2 45^2+\left(\dfrac{3}{5}\right)^2 20^2+(2)\left(\dfrac{2}{5}\right)\left(\dfrac{3}{5}\right)(225)$ $=576$ $\left\lvert\overrightarrow{OP}\right\rvert=24$		

Solution	Marks	Remarks
4. (a) $\int x^2 e^{-x}\mathrm{d}x$		
$= -\int x^2 \mathrm{d}e^{-x}$		
$= -x^2e^{-x} + \int e^{-x}\mathrm{d}x^2$		
$= -x^2e^{-x} + \int 2xe^{-x}\mathrm{d}x$		
$= -x^2e^{-x} - \int 2x\mathrm{d}e^{-x}$		
$= -x^2e^{-x} - 2xe^{-x} + 2\int e^{-x}\mathrm{d}x$		
$= -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + \text{constant}$		
(b) Required Area		
$= \int_0^6 x^2e^{-x}\mathrm{d}x$		
$= \int_0^6 x^2e^{-x}\mathrm{d}x$		
$= \left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^6$		
$= 2 - \dfrac{50}{e^6}$		

Solution	Marks	Remarks

5. (a) (i)

$$\Delta=\begin{vmatrix}1 & 2 & -1\\ 3 & 8 & -11\\ 2 & 3 & h\end{vmatrix}$$

$$=\begin{vmatrix}1 & 2 & -1\\ 3 & 8 & -11\\ 2 & 3 & h\end{vmatrix}$$

$$=\begin{vmatrix}1 & 2 & -1\\ 0 & 2 & -8\\ 0 & -1 & h+2\end{vmatrix}$$

$$=2(h+2)-(-1)(-8)$$

$$=2h+4$$

(E) has unique solution

$\Leftrightarrow\ 2h+4\neq 0$

$\Leftrightarrow\ h\neq -2$

Range of values of h:

$h<-2$ or $h>-2$

(ii)

$$z=\frac{\begin{vmatrix}1 & 2 & 11\\ 3 & 8 & 49\\ 2 & 3 & k\end{vmatrix}}{\Delta}$$

$$=\frac{\begin{vmatrix}1 & 2 & 11\\ 0 & 2 & 16\\ 0 & -1 & k-22\end{vmatrix}}{\Delta}$$

$$=\frac{2k-44+16}{2h-4}$$

$$=\frac{k-16}{h-2}$$

(b) (E) has infinitely solution $\Rightarrow\ h=\text{-}2$

$$\left(\begin{array}{ccc|c}1 & 2 & -1 & 11\\ 3 & 8 & -11 & 49\\ 2 & 3 & 2 & k\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|c}1 & 2 & -1 & 11\\ 0 & 1 & -4 & 8\\ 0 & 0 & 0 & k-14\end{array}\right)$$

(E) has infinitely solution $\Leftrightarrow\ h=\text{-}2$ and $k=14$

$(x,y,z)=(-7t-5,8+4t,t)$ for $t\in\mathbf{R}$

Solution	Marks	Remarks
6. (a) Let r cm be the radius of the water inside the container		
$\frac{r}{15}=\frac{h}{20}$		
$r=\frac{3}{4}h$		
$A=\pi\left(\frac{3}{4}h\right)\sqrt{\left(\frac{3}{4}h\right)^2+h^2}$		
$A=\frac{15}{16}\pi h^2$		
(b) $\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{15}{16}\pi(2h)\frac{\mathrm{d}h}{\mathrm{d}t}$		
When the volume of the container is 96π,		
$96\pi=\frac{1}{3}\pi\left(\frac{3}{4}h\right)^2h$		
$h=8$		
$\left.\frac{\mathrm{d}A}{\mathrm{d}t}\right\vert_{h=8}$		
$=\frac{15}{16}\pi(8)\frac{3}{\pi}$		
$=45$		
The rate of change of the set curved surface = 45 $\mathrm{cm}^2\ \mathrm{s}^{-1}$		

Solution	Marks	Remarks
7. (a) $\sin 3x$ $=\sin 2x\cos x+\cos 2x\sin x$ $=(2\sin x\cos x)\cos x+(1-2\sin^2 x)\sin x$ $=2\sin x(1-\sin^2 x)+(1-2\sin^2 x)\sin x$ $=3\sin x-4\sin^3 x$		
(b) (i) $\dfrac{\sin 3\left(x-\frac{\pi}{4}\right)}{\sin\left(x-\frac{\pi}{4}\right)}$ $=\dfrac{\sin 3x\cos\frac{3\pi}{4}-\sin\frac{3\pi}{4}\cos 3x}{\sin x\cos\frac{\pi}{4}-\cos x\sin\frac{\pi}{4}}$ $=\dfrac{-\sin 3x\cos\left(\pi-\frac{3\pi}{4}\right)-\sin\left(\pi-\frac{3\pi}{4}\right)\cos 3x}{\sin x\cos\frac{\pi}{4}-\cos x\sin\frac{\pi}{4}}$ $=\dfrac{-\frac{\sqrt{2}}{2}\sin 3x-\frac{\sqrt{2}}{2}\cos 3x}{\frac{\sqrt{2}}{2}\sin x-\frac{\sqrt{2}}{2}\cos x}$ $=\dfrac{\sin 3x+\cos 3x}{\cos x-\sin x}$		
(ii) $\dfrac{\sin 3x+\cos 3x}{\cos x-\sin x}=2$ $\dfrac{3\sin\left(x-\frac{\pi}{4}\right)-4\sin^3\left(x-\frac{\pi}{4}\right)}{\sin\left(x-\frac{\pi}{4}\right)}=2$ $3-4\sin^2\left(x-\frac{\pi}{4}\right)=2$ $\sin\left(x-\frac{\pi}{4}\right)=\frac{1}{2}$ $\left(0<\sin\left(x-\frac{\pi}{4}\right)<\frac{\sqrt{2}}{2}\right)$ $x-\frac{\pi}{4}=\frac{\pi}{6}$ $\left(0<x-\frac{\pi}{4}<\frac{\pi}{4}\right)$ $x=\frac{5\pi}{12}$		

Solution	Marks	Remarks
8. (a) The equation of the tangent to Γ at P:		
$\dfrac{y-7}{x-e^3}=\mathrm{f}'(e^3)$		
$\dfrac{y-7}{x-e^3}=\dfrac{1}{e^3}\ln\left(e^3\right)^2$		
$y=\dfrac{6}{e^3}x+1$		
(b) $y=\displaystyle\int\frac{1}{x}\ln x^2\mathrm{d}x$		
$y=\displaystyle\int\frac{2}{x}\ln x\mathrm{d}x$		
$y=2\displaystyle\int\ln x\mathrm{d}\ln x$		
$y=(\ln x)^2+C$, where C is a constant		
Since Γ passes through point $(e^3, 7)$, we have $7=\left(\ln e^3\right)^2+C$		
$C=-2$		
Thus, the equation of Γ is $y=(\ln x)^2-2$		
(c) $\mathrm{f}'(x)=\dfrac{2}{x}\ln x$		
$\mathrm{f}''(x)$		
$=2\left(\dfrac{x\left(\frac{1}{x}\right)-\ln x}{x^2}\right)$		
$=2\left(\dfrac{1-\ln x}{x^2}\right)$		
$\mathrm{f}''(x)=0$		
$2\left(\dfrac{1-\ln x}{x^2}\right)=0$		
$x=e$		
$\mathrm{f}''(x)\begin{cases}>0, & 0<x<e\\ =0, & x=e\\ <0, & x>e\end{cases}$		
The point of inflexion of $\Gamma=(e,-1)$		

Solution	Marks	Remarks
9. (a) $f(x)=x-9+\dfrac{36}{x+4}$ The equation of vertical asymptote : $x=-4$ The equation of oblique asymptote : $y=x-9$		
(b) $f'(x)$ $=\dfrac{(x+4)(2x-5)-(x^2-5x)(1)}{(x+4)^2}$ $=\dfrac{x^2+8x-20}{(x+4)^2}$ $=\dfrac{(x-2)(x+10)}{(x+4)^2}$		
$f'(x)$ $=1-\dfrac{36}{(x+4)^2}$ $=\dfrac{x^2+8x-20}{(x+4)^2}$ $=\dfrac{(x-2)(x+10)}{(x+4)^2}$		
(c) $f'(x)=0 \Leftrightarrow x=2$ or $x=-10$		

x	$(-\infty,-10)$	-10	$(-10,2)$	2	$(2,+\infty)$
$f'(x)$	+	0	–	0	+

The maximum point $=(-10,-25)$

The minimum point $=(2,-1)$

Solution	Marks	Remarks
(d) The volume of the solid		
$= \pi \int_0^5 \left(\frac{x^2 - 5x}{x+4} \right)^2 dx$		
$= \pi \int_0^5 \left(\frac{x^2 - 5x}{x+4} \right)^2 dx$		
$= \pi \int_4^9 \left(\frac{(y-4)^2 - 5(y-4)}{y} \right)^2 d(y+4)$ (Let $y = x+4$)		
$= \pi \int_4^9 \left(\frac{y^2 - 13y + 36}{y} \right)^2 dy$		
$= \pi \int_4^9 \frac{y^4 - 26y^3 + 241y^2 - 936y + 1296}{y^2} dy$		
$= \pi \int_4^9 \frac{y^4 - 26y^3 + 241y^2 - 936y + 1296}{y^2} dy$		
$= \pi \int_4^9 \left(y^2 - 26y + 241 - \frac{936}{y} + \frac{1296}{y^2} \right) dy$		
$= \pi \left[\frac{y^3}{3} - 13y^2 + 241y - 936 \ln\lvert y \rvert - \frac{1296}{y} \right]_4^9$		
$= \pi \left(\frac{2285}{3} - 1872 \ln \frac{3}{2} \right)$		

Solution	Marks	Remarks
10. (a) $\overrightarrow{AC}=6\mathbf{i}-6\mathbf{j}$		
$\overrightarrow{AB}=2\mathbf{i}+\mathbf{j}+\mathbf{k}$		
$\overrightarrow{AE}$ $=\dfrac{\overrightarrow{AC}+r\overrightarrow{AB}}{r+1}$ $=\dfrac{1}{r+1}((2r+6)\mathbf{i}+(r-6)\mathbf{j}+r\mathbf{k})$		
$\overrightarrow{AD}$ $=\dfrac{1}{2}\overrightarrow{AC}$ $=3\mathbf{i}-3\mathbf{j}$		
$\overrightarrow{AE}$ $=\dfrac{10\overrightarrow{AD}+\overrightarrow{AF}}{11}$		
$\overrightarrow{AF}$ $=11\overrightarrow{AE}-10\overrightarrow{AD}$ $=\dfrac{11}{r+1}((2r+6)\mathbf{i}+(r-6)\mathbf{j}+r\mathbf{k})-30\mathbf{i}+30\mathbf{j}$		
$AB\,//\,AF$		
$\dfrac{\frac{11(6+2r)}{r+1}-30}{2}=\dfrac{\frac{11}{r+1}(r-6)}{1}=\dfrac{\frac{11}{r+1}r}{1}$		
$\begin{cases}18-4r=11r\\18-4r=-36+41r\end{cases}$		
$r=\dfrac{6}{5}$ satisfies both equations		
Thus, $r=\dfrac{6}{5}$		

Solution	Marks	Remarks
(b) (i) $\overrightarrow{AE}=\frac{42}{11}\mathbf{i}-\frac{24}{11}\mathbf{j}+\frac{6}{11}\mathbf{k}$ $\overrightarrow{DE}$ $=\overrightarrow{DA}+\overrightarrow{AE}$ $=-3\mathbf{i}+3\mathbf{j}+\frac{42}{11}\mathbf{i}-\frac{24}{11}\mathbf{j}+\frac{6}{11}\mathbf{k}$ $=\frac{9}{11}\mathbf{i}+\frac{9}{11}\mathbf{j}+\frac{6}{11}\mathbf{k}$ $\overrightarrow{AD}\cdot\overrightarrow{DE}$ $=(3\mathbf{i}-3\mathbf{j})\cdot\left(\frac{9}{11}\mathbf{i}+\frac{9}{11}\mathbf{j}+\frac{6}{11}\mathbf{k}\right)$ $=3\left(\frac{9}{11}\right)-3\left(\frac{9}{11}\right)+0\left(\frac{6}{11}\right)$ $=0$		
(ii) $\overrightarrow{AB}\cdot\overrightarrow{BC}$ $=(2\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(4\mathbf{i}-7\mathbf{j}-\mathbf{k})$ $=(2)(4)+(1)(-7)+(1)(-1)$ $=0$ $\angle FBC=90°$ By (b)(i) $\overrightarrow{AD}\cdot\overrightarrow{DE}=0$, we have $\angle FDC=90°$ $\angle FDC=\angle FBC=90°$ Yes, B, D, C, F are concyclic. (converse of $\angle$s in the same segment)		

Solution	Marks	Remarks
(c) Since $\angle FBC = 90°$, Q is the mid-point of CF $\overrightarrow{BF}$ $= \overrightarrow{AF} - \overrightarrow{AB}$ $= 11\overrightarrow{AE} - 10\overrightarrow{AD} - \overrightarrow{AB}$ $= 10\mathbf{i} + 5\mathbf{j} + 5\mathbf{k}$ $\overrightarrow{BQ}$ $= \frac{1}{2}\left(\overrightarrow{BC} + \overrightarrow{BF}\right)$ $= 7\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ $\overrightarrow{AQ}$ $= \overrightarrow{AB} + \overrightarrow{BQ}$ $= 9\mathbf{i} + 3\mathbf{k}$ $\overrightarrow{AP}$ $= \mathbf{i} + 7\mathbf{j} - 2\mathbf{k}$ Volume of the tetrahedron $= \left\|\frac{1}{6}\left(\overrightarrow{AB} \times \overrightarrow{AQ}\right) \cdot \overrightarrow{AP}\right\|$ $= \left\|\frac{1}{6}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 9 & 0 & 3 \end{vmatrix} \cdot \overrightarrow{AP}\right\|$ $= \left\|\frac{1}{6}(3\mathbf{i} + 3\mathbf{j} - 9\mathbf{k}) \cdot (\mathbf{i} + 7\mathbf{j} - 2\mathbf{k})\right\|$ $= 7$		

Solution	Marks	Remarks
11. (a) Let $x=-1+\sqrt{2}\tan\theta$		
So, we have $\mathrm{d}x=\sqrt{2}\sec^2\theta\,\mathrm{d}\theta$		
$\int_0^1\frac{1}{x^2+2x+3}\mathrm{d}x$		
$=\int_0^1\frac{1}{(x+1)^2+2}\mathrm{d}x$		
$=\int_{\tan^{-1}\frac{\sqrt{2}}{2}}^{\tan^{-1}\sqrt{2}}\frac{1}{(-1+\sqrt{2}\tan\theta+1)^2+2}\sqrt{2}\sec^2\theta\,\mathrm{d}\theta$		
$=\int_{\tan^{-1}\frac{\sqrt{2}}{2}}^{\tan^{-1}\sqrt{2}}\frac{1}{2(1+\tan^2\theta)}\sqrt{2}\sec^2\theta\,\mathrm{d}\theta$		
$=\int_{\tan^{-1}\frac{\sqrt{2}}{2}}^{\tan^{-1}\sqrt{2}}\frac{1}{2\sec^2\theta}\sqrt{2}\sec^2\theta\,\mathrm{d}\theta$		
$=\frac{\sqrt{2}}{2}\int_{\tan^{-1}\frac{\sqrt{2}}{2}}^{\tan^{-1}\sqrt{2}}\mathrm{d}\theta$		
$=\frac{\sqrt{2}}{2}[\theta]_{\tan^{-1}\frac{\sqrt{2}}{2}}^{\tan^{-1}\sqrt{2}}$		
$=\frac{\sqrt{2}}{2}\left(\tan^{-1}\sqrt{2}-\tan^{-1}\frac{\sqrt{2}}{2}\right)$		
$=\frac{\sqrt{2}}{2}\tan^{-1}\frac{\sqrt{2}}{4}$		

Solution	Marks	Remarks
(b) (i) Let $t=\tan\theta$ $\cos\theta=\dfrac{1}{\sqrt{1+t^2}}$ $\sin\theta=\dfrac{t}{\sqrt{1+t^2}}$ $\sin 2\theta$ $=2\sin\theta\cos\theta$ $=2\left(\dfrac{t}{\sqrt{1+t^2}}\right)\left(\dfrac{1}{\sqrt{1+t^2}}\right)$ $=\dfrac{2t}{1+t^2}$ $=\dfrac{2\tan\theta}{1+\tan^2\theta}$ $\cos 2\theta$ $=2\cos^2\theta-1$ $=2\left(\dfrac{1}{\sqrt{1+t^2}}\right)^2-1$ $=\dfrac{2}{1+t^2}-1$ $=\dfrac{1-t^2}{1+t^2}$ $=\dfrac{1-\tan^2\theta}{1+\tan^2\theta}$		
(b) (ii) Let $t=\tan\theta$ So, we have $\mathrm{d}t=\sec^2\theta\,\mathrm{d}\theta$ $\mathrm{d}\theta=\dfrac{1}{1+t^2}\mathrm{d}t$ $\int_0^{\frac{\pi}{4}}\dfrac{1}{\sin 2\theta+\cos 2\theta+2}\mathrm{d}\theta$ $=\int_0^1\dfrac{1}{\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}+2}\dfrac{1}{1+t^2}\mathrm{d}t$ $=\int_0^1\dfrac{1}{t^2+2t+3}\mathrm{d}t$ $=\dfrac{\sqrt{2}}{2}\tan^{-1}\dfrac{\sqrt{2}}{4}$		

Solution	Marks	Remarks
11. (c)		

Let $\theta = \frac{\pi}{4} - \phi$

So, we have $\mathrm{d}\theta = -\,\mathrm{d}\phi$

$$\int_0^{\frac{\pi}{4}} \frac{\sin 2\theta + 1}{\sin 2\theta + \cos 2\theta + 2}\,\mathrm{d}\theta$$
$$= \int_{\frac{\pi}{4}}^{0} \frac{\sin 2\left(\frac{\pi}{4} - \phi\right) + 1}{\sin 2\left(\frac{\pi}{4} - \phi\right) + \cos 2\left(\frac{\pi}{4} - \phi\right) + 2}(-1)\,\mathrm{d}\phi$$
$$= \int_0^{\frac{\pi}{4}} \frac{\sin\left(\frac{\pi}{2} - 2\phi\right) + 1}{\sin\left(\frac{\pi}{2} - 2\phi\right) + \cos\left(\frac{\pi}{2} - 2\phi\right) + 2}\,\mathrm{d}\phi$$
$$= \int_0^{\frac{\pi}{4}} \frac{\cos 2\phi + 1}{\sin 2\phi + \cos 2\phi + 2}\,\mathrm{d}\phi$$
$$= \int_0^{\frac{\pi}{4}} \frac{\cos 2\theta + 1}{\sin 2\theta + \cos 2\theta + 2}\,\mathrm{d}\theta$$

(d)

$$\int_0^{\frac{\pi}{4}} \frac{\cos 2\theta + 1}{\sin 2\theta + \cos 2\theta + 2}\,\mathrm{d}\theta + \int_0^{\frac{\pi}{4}} \frac{\sin 2\theta + 1}{\sin 2\theta + \cos 2\theta + 2}\,\mathrm{d}\theta$$
$$= \int_0^{\frac{\pi}{4}} \frac{\sin 2\theta + \cos 2\theta + 2}{\sin 2\theta + \cos 2\theta + 2}\,\mathrm{d}\theta$$
$$= \int_0^{\frac{\pi}{4}} \mathrm{d}\theta$$
$$= \frac{\pi}{4}$$

By (c), $\int_0^{\frac{\pi}{4}} \frac{\sin 2\theta + 1}{\sin 2\theta + \cos 2\theta + 2}\,\mathrm{d}\theta = \int_0^{\frac{\pi}{4}} \frac{\cos 2\theta + 1}{\sin 2\theta + \cos 2\theta + 2}\,\mathrm{d}\theta$

So, we have $\int_0^{\frac{\pi}{4}} \frac{\sin 2\theta + 1}{\sin 2\theta + \cos 2\theta + 2}\,\mathrm{d}\theta = \frac{\pi}{8}$

$$\int_0^{\frac{\pi}{4}} \frac{8\sin 2\theta + 9}{\sin 2\theta + \cos 2\theta + 2}\,\mathrm{d}\theta$$
$$= 8\int_0^{\frac{\pi}{4}} \frac{\sin 2\theta + 1}{\sin 2\theta + \cos 2\theta + 2}\,\mathrm{d}\theta + \int_0^{\frac{\pi}{4}} \frac{1}{\sin 2\theta + \cos 2\theta + 2}\,\mathrm{d}\theta$$
$$= \pi + \frac{\sqrt{2}}{2}\tan^{-1}\frac{\sqrt{2}}{4}$$

Solution	Marks	Remarks
12. (a) Let P(n): $A^n = 3^n I + 3^{n-1} n \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$		
For $n = 1$,		
$3^1 I + 3^0 (1) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$		
$= \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$		
$= \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$		
$= A$		
$\therefore$ P(1) is true		
Assume P(k) is true for some positive integers k,		
i.e. $A^k = 3^k I + 3^{k-1} k \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$		
For $n = k + 1$,		
A^{k+1}		
$= A^k A$		
$= \left(3^k I + 3^{k-1} k \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$		
$= 3^k \begin{pmatrix} 1 & \frac{k}{3} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$		
$= 3^k \begin{pmatrix} 3 & k+1 \\ 0 & 3 \end{pmatrix}$		
$= 3^k \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} + 3^k \begin{pmatrix} 0 & k+1 \\ 0 & 0 \end{pmatrix}$		
$= 3^{k+1} I + 3^{k-1} (k+1) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$		
$\therefore$ P($k+1$) is true		
By M.I., $A^n = 3^n I + 3^{n-1} n \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is true for all positive integers n.		

Solution	Marks	Remarks

12. (b) (i)

$$P^{-1}$$
$$=\begin{pmatrix}-1 & 0\\ -2 & -1\end{pmatrix}$$

$$P^{-1}BP$$
$$=\begin{pmatrix}-1 & 0\\ -2 & -1\end{pmatrix}\begin{pmatrix}5 & 1\\ -4 & 1\end{pmatrix}\begin{pmatrix}-1 & 0\\ 2 & -1\end{pmatrix}$$
$$=\begin{pmatrix}-5 & -1\\ -6 & -3\end{pmatrix}\begin{pmatrix}-1 & 0\\ 2 & -1\end{pmatrix}$$
$$=\begin{pmatrix}3 & 1\\ 0 & 3\end{pmatrix}$$

(ii)

$$P^{-1}BP = A$$
$$P^{-1}B^nP = A^n$$
$$P^{-1}B^nP = 3^n I + 3^{n-1}n\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$$

$$B^n$$
$$=3^n PP^{-1} + 3^{n-1}nP\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}P^{-1}$$
$$=3^n I + 3^{n-1}n\begin{pmatrix}-1 & 0\\ 2 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}-1 & 0\\ -2 & -1\end{pmatrix}$$
$$=3^n I + 3^{n-1}n\begin{pmatrix}0 & -1\\ 0 & 2\end{pmatrix}\begin{pmatrix}-1 & 0\\ -2 & -1\end{pmatrix}$$
$$=3^n I + 3^{n-1}n\begin{pmatrix}2 & 1\\ -4 & -2\end{pmatrix}$$ for any positive integer n

(iii)

$$\left|A^m - B^m\right|$$
$$=\left|3^m I + 3^{m-1}m\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} - 3^m I - 3^{m-1}m\begin{pmatrix}2 & 1\\ -4 & -2\end{pmatrix}\right|$$
$$=\left|3^{m-1}m\begin{pmatrix}-2 & 0\\ 0 & 2\end{pmatrix}\right|$$
$$=\left(3^{m-1}m\right)^2\begin{vmatrix}-2 & 0\\ 0 & 2\end{vmatrix}$$
$$=-4\left(3^{2m-2}\right)m^2$$

Since $3^{2m-2} > 0$ and $m^2 > 0$ for all positive integers m,
So, $\left|A^m - B^m\right| < 0$ for all positive integers m.

However, $4m^2 > 0$ for all positive integers m.
No, there does not exist.