High Performance – Question 10

10. An engineer models the rates of the production of an alloy in the first 10 weeks by two new machines A and B respectively by

$$\frac{dx}{dt}=\frac{61t}{(t+1)^{\frac{5}{2}}} \quad \text{and} \quad \frac{dy}{dt}=\frac{15\ln(t^2+100)}{16} \qquad \text{for } 0\le t\le 10 ,$$

where x (in million kg) and y (in million kg) are the amount of the alloy produced by machines A and B respectively, and t (in weeks) is the time elapsed since the beginning of the production.

(a) Using the substitution $u=t+1$, find the amount of the alloy produced by machine A in the first 10 weeks. (4 marks)

(b) Using the trapezoidal rule with 5 sub-intervals, estimate the amount of the alloy produced by machine B in the first 10 weeks. (2 marks)

(c) The engineer uses the results of (a) and (b) to claim that machine B is more productive than machine A in the first 10 weeks. Do you agree? Explain your answer. (4 marks)

10 (a) Amount by A $= \int_0^{10} \frac{61t}{(t+1)^{5/2}}\,dt$ ✓ 1A

$= \int_1^{11} \frac{61(u-1)}{u^{5/2}}\,du$ $(\because u=t+1)$

$= \int_1^{11} 61[u^{-\frac{3}{2}}-u^{-\frac{5}{2}}]\,du$ ✓ 1M

$= 61\left[\frac{u^{-\frac{1}{2}}}{-\frac{1}{2}}+\frac{u^{-\frac{3}{2}}}{\frac{3}{2}}\right]_1^{11}$ ✓ 1A

$= 45.6636$ ✓ 1A

(b) Amount by B $= \int_0^{10} \frac{15}{16}\ln(t^2+100)\,dt$

$\approx \frac{2}{2}\times\frac{15}{16}[\ln(0^2+100)+2\ln(2^2+100)+2\ln(4^2+100)+2\ln(6^2+100)+2\ln(8^2+100)+\ln(10^2+100)]$ 1M

$= 45.6792$ ✓ 1A

(c) $\frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\frac{15}{16}\ln(t^2+100)$

$= \frac{15}{16}\frac{(2t)}{(t^2+100)}$

$= \frac{15}{8}\frac{t}{(t^2+100)}$ ✓ 1A

$\frac{d^2}{dt^2}\left(\frac{dy}{dt}\right) = \frac{15}{8}\left[\frac{t^2+100-t(2t)}{(t^2+100)^2}\right] = \frac{15}{8}\left(\frac{100-t^2}{(t^2+100)^2}\right)$ ✓ 1A

For $0<t<10$, $\frac{d^2}{dt^2}\left(\frac{dy}{dt}\right)>0$

$\therefore$ The amount produced by B is a under-estimate ✗

$\therefore$ B is more productive than A ✗

Answers written in the margins will not be marked.

High Performance – Question 13

13. In a supermarket, there are two cashier counters: a regular counter and an express counter. It is known that 88% of customers pay at the regular counter. It is found that the waiting time for a customer to pay at the regular counter follows the normal distribution with mean 6.6 minutes and standard deviation 1.2 minutes.

(a) Find the probability that the waiting time for a customer to pay at the regular counter is more than 6 minutes.
(2 marks)

(b) Suppose 12 customers who pay at the regular counter are randomly selected. Assume that their waiting times are independent.

(i) Find the probability that there are more than 10 of the 12 customers each having a waiting time of more than 6 minutes.

(ii) Find the probability that the average waiting time of the 12 customers is more than 6 minutes.
(5 marks)

(c) It is found that the waiting time for a customer to pay at the express counter follows the normal distribution with mean μ minutes and standard deviation 0.8 minutes. It is known that exactly 21.19% of the customers at the regular counter wait less than k minutes, while exactly 3.59% of the customers at the express counter wait more than k minutes.

(i) Find k and μ.

(ii) Two customers are randomly selected. Assume that their waiting times are independent. Given that both of them wait more than μ minutes to pay, find the probability that exactly one of them pays at the regular counter.
(8 marks)

13 a) $X \sim N(6.6, 1.2^2)$

$P(X>6) = P\left(Z > \frac{6-6.6}{1.2}\right)$ ✓ 1M

$= 0.5 + 0.1915$

$= 0.6915$ ✓ 1A

b) (i) $Y \sim B(0.6915, 12)$

$P(Y>10) = C^{12}_{11}(0.6915)^{11}(1-0.6915) + C^{12}_{12}(0.6915)^{12}$ ✓ ~~1M~~ 1M

$= 0.0759$ ✓ 1A

(ii) $\bar{X} \sim N(6.6, 0.12)$ ✓ 1A

$P(\bar{X}>6) = P\left(Z > \frac{6-6.6}{\sqrt{0.12}}\right)$

$= 0.5 + 0.4584$

$= 0.9584$ ✓ 1A

c) (i) $P(X<k) = 0.2119$

$P\left(Z < \frac{k-6.6}{1.2}\right) = 0.2119$ ✓ 1M

$\frac{k-6.6}{1.2} = -0.8$

$k = 5.64$ ✓ 1A

$W \sim N(\mu, 0.8^2)$

$P(W > 5.64) = 0.0359$

$P\left(Z > \frac{5.64-\mu}{0.8}\right) = 0.0359$ ✓ 1M

$\frac{5.64-\mu}{0.8} = 1.8$

$\mu = 4.2$ ✓ 1A

(ii) $P(X>4.2)$

$= P\left(Z > \frac{4.2-6.6}{1.2}\right)$

$= 0.5 + 0.4772$

$= 0.9772$ ✓ 1A

$P(W>4.2)$

$= 0.5$ ($\because$ mean value $= 4.2$)

The required prob

$= \frac{C^2_1(0.88)(0.12)}{(0.9772)(0.5)}$ ✗

$= 0.4323$ ✗

Answers written in the margins will not be marked.

Mid Performance – Question 6

6. A random sample of size 10 is drawn from a normal population with mean μ and variance 8. Let $\overline{X}$ be the mean of the sample.

   (a) Calculate $\mathrm{Var}(2\overline{X}+7)$.

   (b) Suppose the mean of the sample is 50. Construct a 97% confidence interval for μ.

(5 marks)

6 a) $\overline{X} \sim N\left(\mu, \sqrt{\frac{8}{10}}^{2}\right)$

$\mathrm{Var}(2\overline{X}+7)$

$= 2^2 \mathrm{Var}(\overline{X})$

$= 4\left(\frac{8}{10}\right)$ ✓ 1M

$= 3.2$ ✓ 1A

b) A 97% confidence interval for μ

$= \left(50 - 2.17 \times \frac{8}{10}, 50 + 2.17 \times \frac{8}{10}\right)$ ✗ 1M+1A

$= (48.264, 51.736)$ ✗

表現中等 — 第八題

8. 某農場把其出產的雞蛋裝入箱子中，每個箱子盛有 30 隻雞蛋。一隻隨機選取的雞蛋是變壞雞蛋的概率爲 0.04。

(a) 求某個箱子盛有超過 1 隻變壞雞蛋的概率。

(b) 現逐一檢查雞蛋箱子。

(i) 求第 6 個被檢查的箱子爲第 1 個發現盛有超過 1 隻變壞雞蛋的箱子的概率。

(ii) 求在首次發現盛有超過 1 隻變壞雞蛋的箱子時所曾檢查過的箱子數目的期望值。

（7 分）

8 (a) 所求概率

$= 1 - C_0^{30}(1-0.04)^{30} - C_1^{30}(0.04)(1-0.04)^{29}$ ✓ 1M+1M

≈ 0.338820

$= 0.3388$ ✓ 1A

(b)(i) $P = (1-0.338820)^5(0.338820)$ ✓ 1M

≈ 0.0428121 ✓

$= 0.0428$ ✓ 1A

(ii) 期望值 $= \frac{1}{0.0428}$ ✗

≈ 23.364 個 ✗

表現稍遜 — 第一題

1. (a) 展開 $(2x+1)^3$。

   (b) 依 x 的升冪次序展開 e^{-ax} 到含 x^2 的項爲止，其中 a 爲常數。

   (c) 若在 $\dfrac{(2x+1)^3}{e^{ax}}$ 的展式中，x^2 項的係數是 -4，求 a 的值。

（5 分）

1 (a) $(2x+1)^3 = (2x)^3 + 3(2x)^2 + 3(2x) + 1$

$= 8x^3 + 12x^2 + 6x + 1$ ✓ 1A

(b) $e^{-ax} = 1 - ax + \frac{a^2}{2}x^2 + \cdots$ ✓ 1A

(c) $\dfrac{(2x+1)^3}{e^{ax}} = \dfrac{8x^3 + 12x^2 + 6x + 1}{1 + ax + \frac{a^2}{2}x^2 + \cdots}$ ✗

$\therefore x^2$ 的係數是 $= 12(1) + 8 \cdot \frac{1}{a} = -4$ ✗

$a = -\frac{1}{2}$ ✗

表現稍遜 — 第五題

5. 考慮曲線 $C_1 : y = 1 - \frac{e}{e^x}$ 及曲線 $C_2 : y = e^x - e$ 。

(a) 求 C_1 及 C_2 所有交點的 x 坐標。

(b) 求 C_1 及 C_2 所圍成區域的面積。

（5 分）

5 a) $1 - \frac{e}{e^x} = e^x - e$

$-e = e^{2x} - e^x \cdot e - e^2$

$e^{2x} - (e+1)e^x + e = 0$ ✓ 1A

$(e^x - e)(e^x - 1) = 0$

$e^x = e$ 或 1

$x = 1$ 或 0 ✓ 1A

b) $\int_0^1 (e^x - e) - (1 - \frac{e}{e^x})\,dx$ ✓ 1M PP-1

$= \int_0^1 \frac{e^x - e - 1 + e}{e^x}\,dx$

$= \int_0^1 \frac{e^x - 1}{e^x}\,dx$

$= \int_0^1 1 - e^{-x}\,dx$

$= [x + e^{-x}]_0^1$

$= 1.3679$ ✗ 0A