High Performance – Question 10

10. (a) Find $\int xe^{-x^2}\,\mathrm{d}x$.

(b)

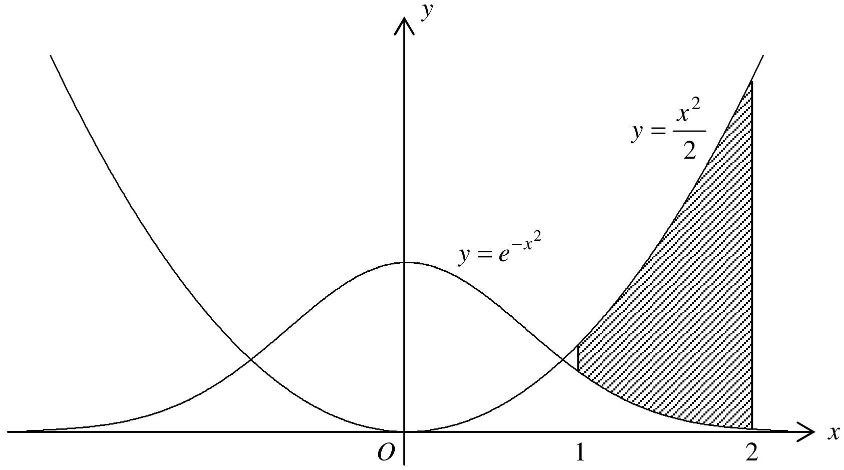

Figure 1

In Figure 1, the shaded region is bounded by the curves $y=\frac{x^2}{2}$ and $y=e^{-x^2}$, where $1\le x\le 2$. Find the volume of the solid generated by revolving the shaded region about the y-axis.

(6 marks)

Start each question o

a) $\int x e^{-x^2} dx$.

$=\frac{1}{2}\int e^{-x^2} d(x^2)$ 1M

$=-\frac{1}{2}\int e^{-x^2} d(-x^2)$

$=-\frac{1}{2}e^{-x^2}+C$ where C is a constant ✓ 1A

b) The volume of revolution

~~$=\int_1^2 2\pi x($~~

~~$=\pi(2)^2 2-\int_1^2 2\pi x(e^{-x^2})dx-\int_1^2 2\pi x(\frac{x^2}{2})dx$~~

~~$=\int^2$~~

✓

$=\int_1^2 2\pi x(\frac{x^2}{2})dx-\int_1^2 2\pi x(e^{-x^2})dx$. 1M+1A

$=\int_1^2 \pi x^3 dx-\int_1^2 2\pi x e^{-x^2}dx$.

$=\frac{\pi x^4}{4}\Big|_1^2+\pi e^{-x^2}\Big|_1^2$ ✓ 1M

$=\left(\frac{2^4\pi}{4}-\frac{\pi}{4}\right)+\left(\pi e^{-4}-\pi e^{-1}\right)$.

$=4\pi-\frac{\pi}{4}+\pi\left(\frac{1}{e^4}-\frac{1}{e}\right)$

$=\frac{15\pi}{16}+\pi\left(\frac{1-e^3}{e^4}\right)$

$=\left(\frac{15}{16}+\frac{1-e^3}{e^4}\right)\pi$

✗

High Performance – Question 13

13. (a) Let $\mathrm{f}(x)$ be an odd function for $-p \le x \le p$, where p is a positive constant.

Prove that $\int_0^{2p} \mathrm{f}(x-p)\,\mathrm{d}x = 0$.

Hence evaluate $\int_0^{2p} [\mathrm{f}(x-p)+q]\,\mathrm{d}x$, where q is a constant.

(4 marks)

(b) Prove that $\dfrac{\sqrt{3}+\tan\left(x-\frac{\pi}{6}\right)}{\sqrt{3}-\tan\left(x-\frac{\pi}{6}\right)} = \dfrac{1+\sqrt{3}\tan x}{2}$.

(2 marks)

(c) Using (a) and (b), or otherwise, evaluate $\int_0^{\frac{\pi}{3}} \ln(1+\sqrt{3}\tan x)\,\mathrm{d}x$.

(4 marks)

13 a. $-f(x) = f(-x)$ ← For odd function,

$\int_0^{2p} f(x-p)\,dx$ (Let $u = x-p$, $du = dx$. When $x = 2p$, $u = p$; When $x = 0$, $u = -p$.) 1M

$= \int_{-p}^{p} f(u)\,du$

$= \int_0^p f(u)\,du + \int_{-p}^{0} f(u)\,du$

$= \int_0^p f(x)\,dx + \int_{-p}^{0} f(x)\,dx$

$= \int_0^p f(x)\,dx - \int_{-p}^{0} f(-x)\,dx$ (Let $t = -x$, $dt = -dx$) 1M

$= \int_0^p f(x)\,dx + \int_p^0 f(t)\,dt$ (When $x = 0$, $t = 0$; When $x = -p$, $t = p$.)

$= \int_0^p f(x)\,dx - \int_0^p f(t)\,dt$

$= \int_0^p f(x)\,dx - \int_0^p f(x)\,dx$ ✓ 1

$= 0$ //

$\int_0^{2p} [f(x-p)+q]\,dx$

$= \int_0^{2p} f(x-p)\,dx + \int_0^{2p} q\,dx$

$= qx \Big|_0^{2p}$

$= 2pq$ // ✓ 1A

b. $\dfrac{\sqrt{3}+\tan(x-\frac{\pi}{6})}{\sqrt{3}-\tan(x-\frac{\pi}{6})} = \dfrac{\sqrt{3}+\frac{\tan x - \tan\frac{\pi}{6}}{1+\tan x\tan\frac{\pi}{6}}}{\sqrt{3}-\frac{\tan x - \tan\frac{\pi}{6}}{1+\tan x\tan\frac{\pi}{6}}}$ 1M

$= \dfrac{\sqrt{3}(1+\frac{\sqrt{3}}{3}\tan x)+\tan x-\frac{\sqrt{3}}{3}}{\sqrt{3}(1+\frac{\sqrt{3}}{3}\tan x)-\tan x+\frac{\sqrt{3}}{3}}$

$= \dfrac{\sqrt{3}+\tan x+\tan x-\frac{\sqrt{3}}{3}}{\sqrt{3}+\tan x-\tan x+\frac{\sqrt{3}}{3}}$

$= \dfrac{2\tan x+\sqrt{3}-\frac{\sqrt{3}}{3}}{\sqrt{3}+\frac{\sqrt{3}}{3}}$

$= \dfrac{6\tan x+3\sqrt{3}-\sqrt{3}}{3\sqrt{3}+\sqrt{3}}$

$= \dfrac{6\tan x+2\sqrt{3}}{4\sqrt{3}}$

$= \dfrac{3\tan x+\sqrt{3}}{2\sqrt{3}}$

$= \dfrac{\sqrt{3}\tan x+1}{2}$ // ✓ 1

c. $\int_0^{\frac{\pi}{3}} \ln(1+\sqrt{3}\tan x)\,dx$

$= \int_0^{\frac{\pi}{3}} \ln\left(\frac{1+\sqrt{3}\tan x}{2}\times 2\right)dx$

$= \int_0^{\frac{\pi}{3}} \left[\ln\left(\frac{1+\sqrt{3}\tan x}{2}\right)+\ln 2\right]dx$

$= \int_0^{\frac{\pi}{3}} \ln\left[\frac{\sqrt{3}+\tan(x-\frac{\pi}{6})}{\sqrt{3}-\tan(x-\frac{\pi}{6})}\right] + \int_0^{\frac{\pi}{3}} \ln 2\,dx$ 1M

$= \int_0^{\frac{\pi}{3}} \ln(\sqrt{3}+\tan(x-\frac{\pi}{6}))\,dx - \int_0^{\frac{\pi}{3}} \ln(\sqrt{3}-\tan(x-\frac{\pi}{6}))\,dx + \int_0^{\frac{\pi}{3}} \ln 2\,dx$

~~$= \int_0^{2(\frac{\pi}{6})} \ln[\sqrt{3}+\tan(x-\frac{\pi}{6})]\,dx - \int_0^{2(\frac{\pi}{6})} \ln(\sqrt{3}-\tan(x-\frac{\pi}{6}))\,dx + \int_0^{\frac{\pi}{3}} \ln 2\,dx$~~

$= \int_0^{\frac{\pi}{3}} \ln 2\,dx$? ? ✗

$= \ln 2 \times \Big|_0^{\frac{\pi}{3}}$

$= \frac{\pi}{3}\ln 2$ // ✓ 1A

表現中等 — 第五題

5. (a) 已知對任意實數 x，$\cos(x+1)+\cos(x-1)=k\cos x$。求 k 的值。

(b) 不用計算機，求 $\begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}$ 的值。

（6 分）

5.(a) $\cos(x+1)+\cos(x-1)=k\cos x$

$2\cos\frac{(x+1)+(x-1)}{2}\cos\frac{(x+1)-(x-1)}{2}=k\cos x$ ✓ 1M

$2\cos\frac{2x}{2}\cos\frac{2}{2}=k\cos x$

$2\cos x\cos(1)=k\cos x$

$2\cos(1)=k$ ✓ 1A

$k=2.000$ ✗

(b) $\begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix} = \cos 1(\cos 5\cos 9-\cos 6\cos 8)$
$-\cos 2(\cos 4\cos 9-\cos 6\cos 7)$
$+\cos 3(\cos 4\cos 8-\cos 5\cos 7)$

$=\frac{\cos 1}{2}\left[(\cos 14+\cos 4)-(\cos 14+\cos 2)\right]$ 1M

$-\frac{\cos 2}{2}\left[(\cos 13+\cos 5)-(\cos 13+\cos 1)\right]$

$+\frac{\cos 3}{2}\left[(\cos 12+\cos 4)-(\cos 12+\cos 2)\right]$

$=\frac{\cos 1}{2}(\cos 4-\cos 2)-\frac{\cos 2}{2}(\cos 5-\cos 1)$

$+\frac{\cos 3}{2}(\cos 4-\cos 2)$ ✗

$=\frac{(\cos 4-\cos 2)(\cos 1+\cos 3)}{2}$

寫於邊界以外的答案，將不予評閱。

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Mid Performance – Question 12

12.

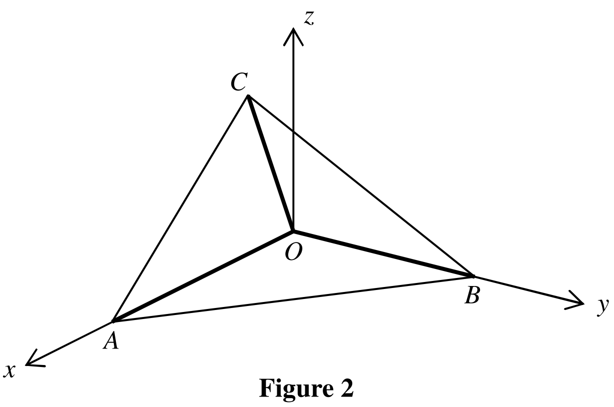

Figure 2

Let $\overrightarrow{OA} = \mathbf{i}$, $\overrightarrow{OB} = \mathbf{j}$ and $\overrightarrow{OC} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ (see Figure 2). Let M and N be points on the straight lines AB and OC respectively such that $AM : MB = a : (1-a)$ and $ON : NC = b : (1-b)$, where $0 < a < 1$ and $0 < b < 1$. Suppose that MN is perpendicular to both AB and OC .

(a) (i) Show that $\overrightarrow{MN} = (a+b-1)\mathbf{i} + (b-a)\mathbf{j} + b\mathbf{k}$.

(ii) Find the values of a and b .

(iii) Find the shortest distance between the straight lines AB and OC .

(8 marks)

(b) (i) Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

(ii) Let G be the projection of O on the plane ABC , find the coordinates of the intersecting point of the two straight lines OG and MN .

(5 marks)

a $\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM}$

$= (\vec{i} + \vec{j} + \vec{k})b - \dfrac{(1-a)(\vec{i}) + (a)\vec{j}}{1-a+a}$ 1M

$= b\vec{i} + b\vec{j} + b\vec{k} - (1-a)\vec{i} - a\vec{j}$

$= (a+b-1)\vec{i} + (b-a)\vec{j} + b\vec{k}$ // ✓ 1

(ii) ∵ $\overrightarrow{MN} \perp$ to both $\overrightarrow{AB}$ and $\overrightarrow{OC}$.

$\overrightarrow{MN} \cdot \overrightarrow{AB} = 0$ and $\overrightarrow{MN} \cdot \overrightarrow{OC} = 0$ 1M+1M

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$
$= \vec{j} - \vec{i}$

$[(a+b-1)\vec{i} + (b-a)\vec{j} + b\vec{k}] \cdot [\vec{j} - \vec{i}] = 0$ — ①

$[(a+b-1)\vec{i} + (b-a)\vec{j} + b\vec{k}] \cdot [\vec{i} + \vec{j} + \vec{k}] = 0$

① ∴ $1 - a - \cancel{b} + \cancel{b} - a = 0$

$-2a = -1$

$a = \frac{1}{2}$ // ✓ 1A

② $a + b - 1 + b - a + b = 0$

$3b - 1 = 0$

$b = \frac{1}{3}$ // ✓ 1A

(iii) The distance is $\overrightarrow{OM}$

$\left|(1-\frac{1}{2})\vec{i} + \frac{1}{2}\vec{j}\right|$

$= \left|\frac{1}{2}\vec{i} + \frac{1}{2}\vec{j}\right|$

$= \sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2}$

$= 5\cancel{4}$ ✗

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$
$= \vec{j} + \vec{k}$

b i $\overrightarrow{AB} \times \overrightarrow{AC}$

$= (-\vec{i} + \vec{j}) \times (\vec{j} + \vec{k})$

$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}$

$= \vec{i} + \vec{j} - \vec{k}$ // ✓ 1A

表現稍遜 — 第三題

3. 以數學歸納法，證明對所有正整數 n，$4^n+15n-1$ 可被 9 整除。

（5 分）

3. 設命題為 P(n)。

當 n = 1 時

$4^{(1)}+15(1)-1 = 18$（它能整除9）

∴ P(1) 為真

即 $4^k+15(k)-1 = 9M$（M 為正整數）

假設 P(k) 為真。

$4^k+15(k)-1 = 9M$

當 ~~k=~~ n = k+1 时，

$4^{k+1}+15(k+1)-1$

$= 4^k+4+15k+15-1-1+1$

$= 4^k+15k+4+15-1-1+1$

$= 4^k+15k-1+4+15-1+1$

$= 9M+18+9\times\frac{1}{9}$

$= 9(M+2)+9\times\frac{1}{9}$

$= 9(M+2+\frac{1}{9})$

$= 9(M+\frac{19}{9})$（它能整除9）

∴ P(2) 為真

∴ 利用數學归纳法 P(n) 為真，对于所有正整數 n，$4^n+15n-1$ 可被 9 整除。

表現稍遜 — 第一題

1. 求在 $(2-x)^9$ 的展式中 x^5 項的係數。

（4 分）

1. $(2-x)^9$

$=[C_0^9 2^9 + C_1^9 2^8 x + C_2^9 2^7 x^2 + C_3^9 2^6 x^3 + C_4^9 2^5 x^4 + C_5^9 2^4 x^5 + \ldots]$ 1M

$=[2^9 + 9(2^8)x + 36(2^7)x^2 + 84(2^6)x^3 + 126(2^5)x^4 + 126(2^4)x^5 + \ldots]$

$=[512 + 2304x + 4608x^2 + 5376x^3 + 4032x^4 + 2016x^5 + \ldots]$ //